Overcoming visual obstacles with the aid of the Supposer

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OVERCOMING VISUAL OBSTACLES WITH THE AID OF THE SUPPOSER*

ABSTRACT. This paper presents research about the effect of intensive work with diagrams on high school students' use of diagrams in geometry. It identifies three obstacles, culled from previous research, which students must overcome when examining and interpreting diagrams: Diagrams are particular; common usage confuses certain standard diagrams with the classes of objects to which they belong; and a single diagram can be viewed and described in different ways. The paper outlines the resources of the Geometric Supposer, a set of microcomputer software tools which were designed to aid students in overcoming these obstacles. The bulk of the paper presents evidence from students' papers and classroom comments indicating that over the course of a year's work students using the Supposer became more facile in their use of diagrams and were able to overcome each of the three obstacles.

INTRODUCTION

Diagrams are one of the most common ways to represent and communicate geometrical knowledge. Yet, despite their benefits, researchers have pointed out that diagrams also present "obstacles children have to overcome in learning about geometry" (Bishop, 1986, p. 150). Presmeg (1986a, 1986b) calls these obstacles "difficulties experienced by visualizers," while Hoz (1981) describes the "rigidity" that arises when students' conceptions are limited by the use of diagrams or mental images. These obstacles can be grouped around three themes: diagrams are particular; common usage confuses certain standard diagrams with the classes of objects to which they belong; and a single diagram can be viewed and described in different ways.

The Particularity of Diagrams

In most cases in a high school geometry class, diagrams are intended as models. They are meant to be understood as representing a class of objects, or as Rissland (1981) says, they are supposed to "contain the essence of a situation" (p. 56). Nevertheless, every diagram has characteristics that are individual and not representative of the class. For example, Rissland points out that a specific acute scalene triangle *ABC* which is meant to represent all triangles "is by no means a universally valid representation since it does not depict obtuse angles" (p. 58).

Presmeg (1986a) indicates that this obstacle causes students to be trapped by "the one-case concreteness of an image or diagram (which) may

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tie thought to irrelevant details, or may even introduce false data" (p. 44). For example, some of her students assumed that lines were parallel if they looked parallel. The particularity of diagrams is also related to students' difficulties with loci (see Schoenfeld, 1986), for by definition a locus is a collection of points; it can be thought of as a result of the superposition of many diagrams, or as alterations to a single diagram. When describing a locus, one must transcend the particular diagram and describe the path that a particular element of the diagram would trace out were some characteristic of the original diagram to be changed continuously.

Standard Diagrams as Models

If students learn a definition only when examining "standard" diagrams, the particularity of diagrams can lead to another obstacle: "an image of a standard figure (diagram) may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram" (Presmeg, 1986a). Students' definitions may include an irrelevant characteristic of the standard diagram, causing difficulties in creating or interpreting diagrams. Hershkowitz (1987a, 1987b) presents evidence of this kind of obstacle. Interestingly,

1. Teachers and students were unwilling to draw the exterior altitudes for obtuse triangles.
2. Teachers and students drew only interior diagonals from the vertices of concave polygons.
3. Teachers and students were much better at recognizing right triangles in an upright position, than when the right angle was "at the top."
4. Teachers and students were much better at recognizing isosceles triangles "standing on their base," than ones that were rotated.

Inability to "See" a Diagram in Different Ways

Gestalt psychologists identified diagrams and pictures which some people are able to reorganize and "see" in different ways. For example, examine Figure 1, the famous "Greek vase" picture. In fact, as Bishop (1986) points

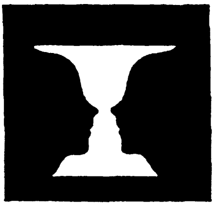


Fig. 1. A picture which can be seen in different ways.

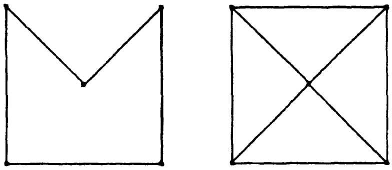


Fig. 2. Figures *A* and *B* from Bishop (1986) p. 152.

out, psychologists often test spatial ability by designing tests like "the classic test of embedded figures . . . where a simple figure, like *A*, must be identified in a more complex figure, like *B*" (p. 152)[1] (see Fig. 2).

Hofstadter (1980) considers this sort of reorganization, sometimes called "figure-ground reversal", to be a central aspect of mathematical creativity. Max Wertheimer (1945) provides examples of situations where students are asked to do this sort of reorganization in high school geometry.[2] Unfortunately, the ability to attend selectively and sequentially to parts and whole does not come easily for many students. According to Hoffer's (1981) formulation, the van Hiele stages suggest that at Level 1 (Recognition) "the student . . . recognizes a shape as a whole." (p. 13) It is only at level 2 (Analysis) that the student can focus on parts of a diagram and analyze properties of figures. As Hoz (1981) also points out, in Figure 3, students may not be able to see *AD* as a side of triangles *ABD* and *ACD* because it is seen only as the "height" of triangle *ABC*. (This difficulty is related to the concept of "functional fixedness" as defined by Anderson, 1985, p. 224.)

These three obstacles were part of the impetus to create The Geometric Supposer[3] which attempts to reduce students' dependence on single diagrams presented in their geometry texts as models for classes of diagrams.[4] Use of the Supposer has been integrated throughout typical year-long Euclidean geometry courses in public high schools to support a style of teaching which includes laboratory sessions where students investigate empirically the characteristics of geometric constructions. In a lab session, students are typically given a problem and asked to record on paper

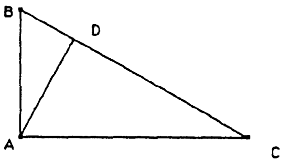


Fig. 3. Right triangle with altitude from right angle vertex.

numerical and visual information gleaned from the Supposer, conjectures about this data, and arguments or proofs to support the conjectures, as well as any other relevant thoughts, questions, or concerns (for an elaborated description of the approach to teaching with the Supposer used in these studies, see Chazan and Houde, 1989, or Yerushalmy *et al.*, 1987).

This pedagogical approach has been used in American public high school Euclidean geometry courses since the 1983–84 school year. For the last four of those years (Years 2 through 5), research has been carried out in Supposer classes. Some of this research studied Supposer and comparison classes for a full year: in 1984–85 (Year 2), two Boston-area Supposer classes and a comparison class (Yerushalmy 1986), in 1985–86 (Year 3), three Boston-area Supposer and three comparison classes (Yerushalmy *et al.*, 1987). Other year-long studies were not comparative. In Israel in 1987–88 (Year 5), two eight-grade (Yerushalmy and Shterenberg, in prep.) and one ninth-grade (Yerushalmy and Maman, 1988) Supposer classes were studied. Finally, some of the research studied smaller instructional units (typically of about one month in duration): in 1986–87 (Year 4), four Boston-area Supposer classes doing a unit on similarity (Chazan, 1988) and two doing a unit on deductive proof, in 1987–88 (Year 5), five Boston-area Supposer classrooms doing a unit on proof (Chazan, 1989).

In the early research projects, both the participating teachers and researchers felt that one of the important differences between students learning with the Supposer and students that these individuals had taught previously involved students' use of diagrams (Yerushalmy *et al.*, 1987). This feeling led to an examination of the collected sources of data, student papers, classroom observations, and paper and pencil tests, for confirmation or disconfirmation. Thus, with respect to the early research projects, the analysis was post-hoc; it was performed to test researchers' and teachers' interpretations of their experience and was not, as in a controlled study, previously planned.

This paper integrates the findings of all of the studies which relate to students' use of diagrams. The evidence from students' papers is presented to convey the richness of the behaviors that were observed in the Supposer classrooms and to argue that some of the students using the Supposer overcame obstacles to geometric learning that many students in previous research projects or in traditional classroom settings do not overcome. The statistical evidence is presented to argue that on average students using the Supposer overcome these obstacles more readily than students in traditional instruction.

In order to suggest that it is plausible for the Supposer to have an effect on students' appreciation of diagrams, before launching into the evidence, the paper continues with a section on the design of the Supposer, especially of the aspects which were designed to help students use diagrams.

THE RESOURCES OF A TEACHING AND LEARNING TOOL

The Supposer was created to aid students in conjecturing and thus to enable teachers to use students' conjecturing to teach high school geometry. The software facilitates the process of making and testing conjectures by generating requested numerical and visual empirical information about geometrical constructions specified by users. Students using this tool are presented with large amounts of visual information (more than in traditional classes). To appreciate this visual information, it is crucial that they overcome the obstacles associated with diagrams. Some of the Supposer's options were explicitly designed to help them do so. This section analyzes how the Supposer's options accomplish this goal; it will examine options separately and discuss the way they interact. This analysis is difficult, because it requires isolating specific software options from the whole, while in reality the impact of the software is the combined impact of all the options. For the ease of the reader, all concrete examples of the workings of options are described as implemented in the Supposer Triangles program for the Apple II+ computer (as opposed to a more recent IBM PC version). The Apple version of the Supposer was the one used by the classes in the research described in this paper, though they used the Quadrilaterals and Circles programs in addition to the Triangles one.

Choosing Initial Shapes

The Supposer makes the classification of shapes into types a salient aspect of students' learning. For example, when using the Triangles program, from the first key press the user must specify the triangle that will be the initial shape. The user can choose to work on a RIGHT, ACUTE scalene, OBTUSE scalene, ISOSCELES, EQUILATERAL, or YOUR OWN triangle. When the user chooses one of the predefined categories, the Supposer challenges the notion of a standard triangle by presenting a random triangle of that kind (random size and where possible random relationship between sides and angles) in a random orientation. Moreover, since each computer boots separately, each screen shows a visibly different triangle; students are confronted with the fact that their diagram is not the only

diagram which can result from the procedure described in a particular problem.

The Supposer also allows the user to test conjectures by creating extreme cases that are candidates for counterexamples. If the user does not choose one of the predefined categories, he or she can exercise options to control the creation of the initial shapes by specifying the size of the sides and angles.

Construction Tools

By providing construction tools that reduce the "overhead" for creating accurate Euclidean geometrical constructions, the Supposer makes it possible to teach and learn based on large quantities of visual information. The Supposer doesn't simply provide an electronic straightedge and compass. Instead the DRAW, LABEL, and ERASE menus include construction tools, like MEDIAN, which can be constructed with compass and straightedge. These construction tools allow the user to create any Euclidean construction quickly and simply (without allowing non-Euclidean constructions).

Furthermore, in the Supposer's menu-driven environment, the user specifies the desired construction by using correct, formal geometric language to describe, without ambiguity, where the construction is to be carried out. For example, in the Triangles program, to create a MEDIAN one must specify the triangle in which, and the vertex from which, the median is to be drawn. There are other ways to create a software environment to do the same task. For example, the user could indicate points, segments, circles, and lines by using a mouse or pointers (as in the CABRI Géomètre software, Laboratoire Structures Discrètes et Didactique, 1988). The precise identification demanded by the Supposer requires that the user pay attention to the diagram and its labels and the labels help the user remember the properties of the different points. (This is also relevant to the REPEAT option described later.)[5]

Yet, the format of the construction tools has other ramifications. For example, within this format, it is not possible to make a mathematically incorrect geometric construction, although it is possible to get an unexpected diagram without making an incorrect key press. If the user has an incorrect, or limited, mental model of a geometric concept, then a correct construction may be surprising. For example, a student might think that a median bisects the vertex angle in addition to the opposite side. For a student with this conception, the diagram shown in Figure 4 might be a surprise.

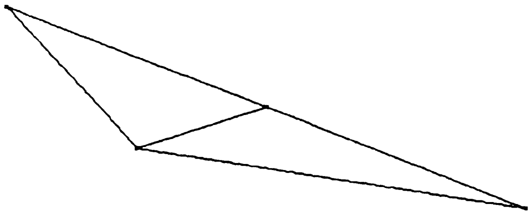


Fig. 4. A median that is clearly not an angle bisector.

Repeating Procedures

The Supposer includes a REPEAT option which also contributes to the ease of the creation of visual information by reducing the construction burden even further. Since the user has unambiguously specified the constructions in formal, geometric language, the Supposer can capture all of the constructions carried out on an initial triangle as a procedure. In the Triangles program, the REPEAT option allows the user to try this procedure on a NEW or PREVIOUS initial triangle. The option to repeat on a previous initial triangle allows the user to move back and forth between four diagrams that result from carrying out the procedure on different initial triangles. Beyond the facility with which diagrams can be created, this option allows users to test their mental images across many cases (freeing users from single diagrams) and to track characteristics of a construction that are invariant from triangle to triangle. For example, a user can draw all three medians in a triangle. Noticing that the three medians are concurrent in this triangle, the user might wonder whether this is true for other triangles. The REPEAT option makes it easy to explore this possibility and thus can lead to a general conjecture which might be proven deductively.

The YOUR OWN option for creating initial shapes also combines nicely with the REPEAT option to allow users to create (invent) their own class of shapes – e.g., the class of triangles whose sides are consecutive numbers (5, 6, 7; 6, 7, 8). Furthermore, this option can be used in a sophisticated manner by users who want their diagrams to be "dynamic." If a user has a variable in mind, for ecample the measure of the largest angle in a triangle, the YOUR OWN and REPEAT options can be used to create a set of initial triangles which "animate" the construction in a systematic manner; the user can track the changes in the diagrams resulting from the procedure as the largest angle of the triangle is incremented.

Complementary Design Choices

The Supposer has many options that are not directly connected with visual information (for example, the measurement utilities). Within these are small design choices that complement the options described so far by helping the user work with visual information. First, when a procedure is repeated on a new initial triangle, the procedure is carried out one construction at a time. The user must press the space bar to continue. This feature helps the user keep track of the procedure and focus on critical aspects of the resulting diagrams. Second, an ERASE facility allows the user to simplify a diagram and focus on a key part. Third, a SCALE utility allows the user to work on the largest diagram which fits on the screen and includes all of the points in the construction. By toggling back and forth between this OPTIMAL diagram and the diagram in the ORIGINAL scale, the user can focus on specific areas of interest or on the whole diagram.

In closing this section, it is extremely important to emphasize that the Supposer does not stand alone; it is part of an approach to teaching geometry that is used by teachers as they see fit and that includes problems and projects for students. The students' work with the software is a part of the course, not the whole. Therefore, as important or even more important than the software itself is how its use is integrated into the course and how teachers make use of the capabilities the software provides.

THE EVIDENCE

Each description of student work is labeled with the year and the type of class in which the research was carried out. Descriptions from the Year 3 study use the labels Techtown, Countrytown, and Rivertown to distinguish the three classes from that study. (For more details about the particular classes and students, see Yerushalmy *et al.*, 1987.)

An initial indication that diagrams had taken on increased importance for Supposer students came from the Year 2 study of two Supposer classes and one comparison class (Yerushalmy, 1986). Statistical results from a generalization test given at the end of the year indicated that most of the non-Supposer students did not use diagrams on the pre or posttests, while on the posttest, "Diagrams seemed to accompany the thinking process of group A (Supposer) students . . . there were numerous free-hand drawings" (Yerushalmy, 1986, pp. 95–96).

Year 5 research in two-eighth grade Israeli Supposer classrooms, using an instrument developed and normed by Hershkowitz (1987a, 1987b),

provides further confirmation; Supposer students are confused less often by diagrams that are in a non-standard orientation. 90.2 percent of the Supposer students successfully drew the altitudes in right triangles of different orientations, as opposed to 28 percent in Hershkowitz's study of 129 eighth-graders (1987b, p. 22); 87.7 percent of the Supposer students successfully drew altitudes in obtuse triangles of different orientations, as opposed to 38 percent for Hershkowitz's nondisadvantaged students and 28 percent for her complete sample (p. 24) (Yerushalmy and Shterenberg, 1989). The remainder of this section presents further evidence grouped under three headings: working with sequences of pictures, reorganizing the visual field, and adding auxiliary lines.

1. *Working with sequences of pictures.* One indication of growth in Supposer students' approach to diagrams was that they stopped identifying whole classes of figures with single diagrams (members of the class). Instead, students worked with many diagrams, saw them as instances of a single class, and were able to abstract the features which characterize the class, leaving behind the particular, non-characteristic aspects of the individual diagrams. In some cases, in class or at the computer, when students saw a series of diagrams representing a class of figures, students envisioned an order, or progression, having an animated, dynamic quality. For example, when the problem presented in Figure 5 was given to the Year 3 Techtown class mid-way through the year, instead of seeing each of the four pictures as separate cases, students described the cases presented in the problem as different pictures of the same situation. In their words, "the point (D) is moving on the circle."

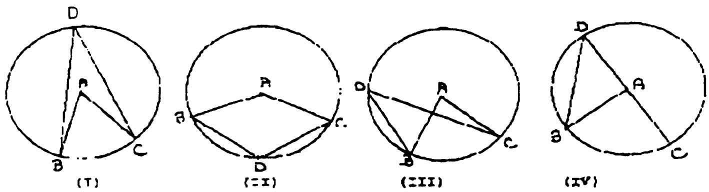
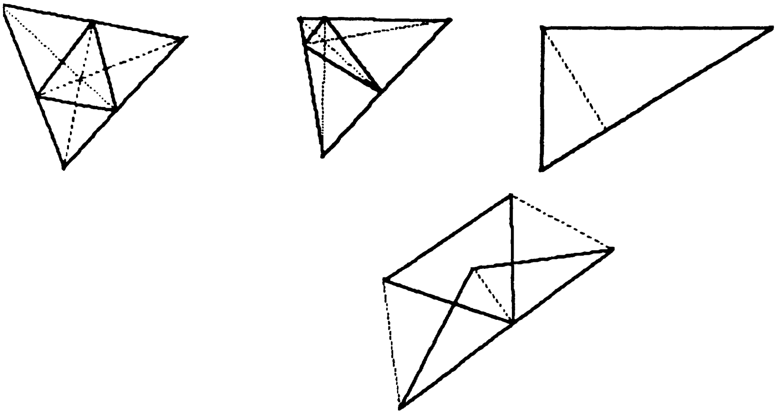


Fig. 5. A circles problem. BC is a minor arc of circle A and D is a point on the circle.

1. Make a drawing that looks like #1 and state a conjecture about the relationship between $m\ BC$ and $m\ BDC$.
2. Make a drawing like #2, #3, and #4 and check that the data supports your conjecture.
3. In *one case* you will find that the data *does not* support your conjecture. Can you find a reasonable explanation for this? Can you *restate* the problem so that this case "works?"
4. Give a proof of your conjecture. (Hint: join D to A and extend the segment DA, so as to form exterior angles for DAB and DAC.)

Many students in the Supposer classes imputed movement to diagrams, treated individual diagrams as snapshots of a process occurring to one underlying configuration, and thought of those snapshots as a class of figures with common characteristics.[6] Beyond "seeing" movement in a set of simultaneously presented diagrams members of the same class, as in the previous example, students saw such movement in examples produced sequentially by the Supposer, created their own displays to show others what they had "seen," and, when working without the computer on a single member of a class, were even able to "imagine" other members of that class and build arguments on this basis.

For example, when students in the Year 3 Countrytown class were asked to draw the three altitudes of a triangle and to connect the feet of the altitudes, they recorded the diagrams which represent the repetition of this construction on different types of initial triangles in order of the largest angle in the triangle (from acute through right and to obtuse) (see Fig. 6).



A student's explanation: why the triangle disappears.

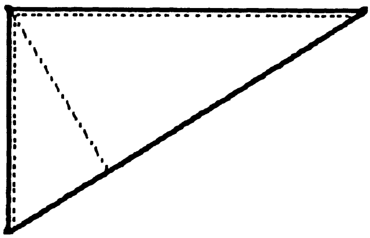


Fig. 6. Three altitudes in a series of triangles.

One pair conjectured that "in a right triangle, if you draw the altitudes, then the center triangle disappears because the altitude is the leg" (Yerushalmy *et al.*, 1986, p. 188). To support their argument the students drew the last diagram shown in Figure 6. Note that their conjecture uses the active verb "disappears" to refer to the triangle that is visible when the construction is made on non-right triangles and the order of their presentation of the diagrams suggests the movement that they describe. Furthermore, their statement indicates that the right triangle diagram is a member of the class of diagrams where "you draw the altitudes."

In the previous example, the structure of the software (the REPEAT option and the classification scheme for triangles) and possibly the demands of the problem help students treat the different diagrams as members of the same class. Yet, there were also instances of students using their own schemes to create classes of diagrams by using the REPEAT option in combination with the YOUR OWN option. For example, students in the Year 3 Rivertown class were asked to construct a triangle by connecting the midpoints of the sides of another triangle and to investigate the ratio between the radii of the circles circumscribed around each of the two triangles. In their exploration, the students created a sequence of diagrams (see Fig. 7) and conjectured that "the triangles with the largest angles have the largest radius ratio (taking the radius of the circumscribed circle/the radius of the inscribed circle). Triangles like the equilateral with all congruent angles have the smallest ratio (2/1)." It is important to note that neither the problem nor the software suggested to students how to "see" these diagrams as a sequence. The students themselves decided to focus on the size of the largest angle as the crucial

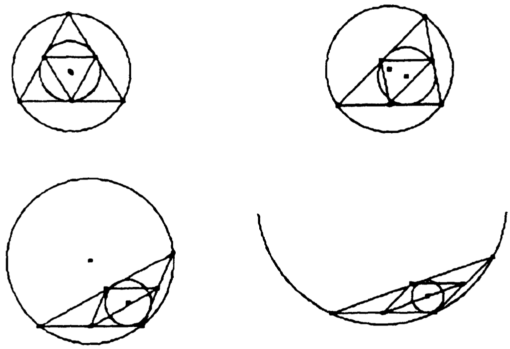


Fig. 7. Getting smaller and smaller.

variable. They explored a relationship that interested them systematically by examining a series of specific members of a class of diagrams.

Students were able to treat a single diagram as a model for a whole class of diagrams and, at the same time, understand that such models may include characteristics not shared by all members of the set. They were able to imagine other members of the set to see if a characteristic seemed common to all. For example, Yerushalmy *et al.* (1987) report:

The class was working with a drawing which included two parallel lines, *BC* and *EF* (see Fig. 8). The students were not told that the lines were parallel, had not yet conjectured that they were, and, with the knowledge they had, could not prove that these lines were parallel. It was however a legitimate and verifiable conjecture. *AD* was an altitude and thus perpendicular to *BC*. One student argued that *AD* was also perpendicular to *EF*, implying that *EF* and *BC* were parallel, but not mentioning that fact. Another student argued that "it doesn't work if *EF* is tilted." In other words, it doesn't work if *EF* is not parallel to *BC*. This assertion was not based on the way the diagram looked. The student was manipulating the diagram and considering an alternative in her "mind's eye." (pp. 15–16)

Clearly, this student understood that a diagram may include characteristics not shared by all members of the class and was able to envision mentally other possible configurations of the diagram.

2. *Reorganizing the visual field.* When presenting his heuristics for understanding a problem, Polya (1945/1973) suggests:

Consider your problem from various sides. Emphasize different parts, examine different details, examine the same details repeatedly, but in different ways, combine the details differently, approach them from different sides. Try to see some new meaning in each detail, some new interpretation of the whole. (p. 34)

In geometry, this exhortation for flexibility translates into the ability to examine a diagram "from various sides" and in different ways. Supposer students' approach to individual diagrams was flexible in the way that

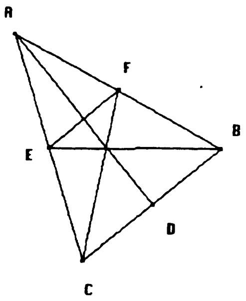


Fig. 8. A construction with a parallel line.

Polya describes; they were able to "see" and focus on different parts of individual diagrams. Testing results, classroom observation, and students' papers indicate that they were willing and able to change their point of view. Some students even created systematic strategies for using the software or for recording conjectures to help change their focus in examining a diagram.

The results of a Year 3 comparison of Supposer and non-Supposer students on a pencil and paper test to evaluate students' conjecturing ability suggest that Supposer students were more likely to write conjectures based on a change in their view of the diagram (Yerushalmy *et al.*, 1987). In some cases students looked at the diagrams differently from the way their teachers intended them to, making it clear that the new "view" had come about without explicit teacher intervention. For example, a teacher in the Year 4 similarity study asked students to draw a right triangle with its interior altitude and then to reflect that altitude over each of the two legs resulting in a shape like shown in Figure 9. As the teacher had hoped, the students conjectured that all five triangles (*BAC*, *AFC*, *AEB*, *ADB*, *ADC*) were similar. In addition, many students focused on aspects of the diagram that the teacher had not considered and produced conjectures which surprised her. Some students focused on the outer segments of the diagram and conjectured that the outside shape was a right trapezoid, while others focused on one vertex (not even a closed shape) and conjectured that angles *FAC*, *CAB*, and *BAE* made a straight line. Both of these conjectures are based on a selective attention to the details of the diagram; the students focused on a part of the diagram and lifted it from its context.

When students were asked to draw all three angle bisectors in an isosceles triangle and label the point of intersection (Fig. 10), a pair of students turned in the diagrams presented in Figure 10 accompanying their

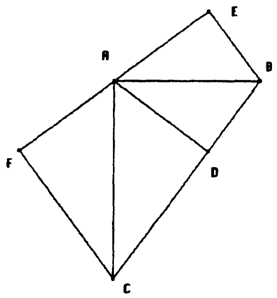


Fig. 9. A right trapezoid.

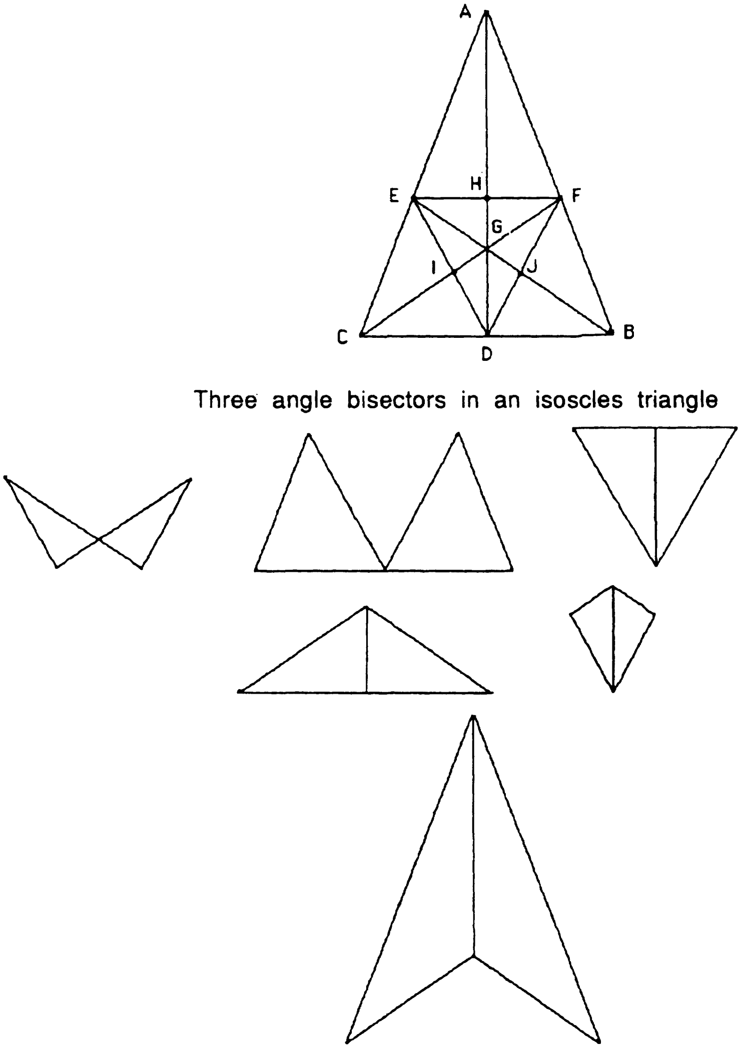


Fig. 10. Parts of the orginal diagram.

conjectures and proofs. By making these diagrams separately from their recording of the initial diagram, these students indicated which part of the diagram they were focussing on and presented their conjectures clearly for their teacher.

While the students whose work appears in Figure 10 used "peeling away" strategies,[7] other students in their class used "building up" strategies:

Instead of completing the whole construction and then repeating the construction on different initial triangles, a pair of students first added one new element to their triangle, made conjectures, and when necessary repeated the construction before going on to the next step of the construction. They explained that this method would enable them to "see" things that they might otherwise miss as they examined the figure anew after each additional step. Their method is remarkable also for its emphasis on the process of making the construction and not the final product, the completed diagram.

3. *Adding auxiliary lines.* Students who add auxiliary lines to diagrams indicate by their action that for them diagrams are not an untouchable, final product, but rather the result of a process which can be added to or changed. Polya (1954) presents several problem situations in which the construction and use of auxiliary lines indicates an understanding, or learning, of geometry that transcends having been merely "taught" geometry. In Polya's view, adding auxiliary lines helps the problem solver access prior knowledge. "Having recollected a formerly solved related problem and wishing to use it for our present one, we must often ask: Should we introduce some auxiliary element in order to make its use possible?" (p. 47). Auxiliary lines have yet another function, a creative one. Adding an auxiliary line, and focusing on parts of the diagram which include this new line, can generate new insights into the diagram, both questions and conjectures. Supposer students used auxiliary lines both on and off the computer in service of proof and to create new conjectures.

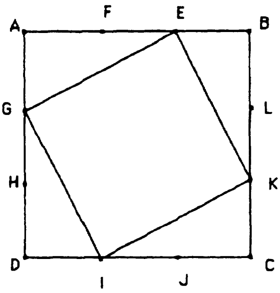
During Year 2 (Yerushalmy, 1986), as part of a paper-and-pencil test on proof skills, both Supposer and non-Supposer students were asked to provide arguments explaining why the unfamiliar statement that "In a regular 8-gon the sum of the 8 exterior angles is 360 degrees" is true. The Supposer students used auxiliary lines to look for previously learned connected facts. "Statistical analysis of the results showed that students in the inductive (Supposer) class used analysis by diagram (adding auxiliary lines) as their main tool for reaching a convincing argument significantly more often than students in the traditional (non-Supposer) class" (p. 191) (see Yerushalmy, 1986, pp. 191–194 for four individual cases of Supposer students' addition of auxiliary lines). A similar test was given to Year 3 Supposer and non-Supposer students with similar results (Yerushalmy *et al.*, 1987). This use of auxiliary lines in service of proof was observed throughout the second half of the course in Supposer classes (after students had been introduced to deductive proof); it did not appear out of the blue

at the end of the year. Frequently, the episodes of adding auxiliary lines occurred in the computer lab as students looked for supporting arguments for conjectures they had developed from their empirical work. (For a full exposition of such an example, see Yerushalmy, 1986, pp. 203–5.)

Supposer students developed creative, irregular, or surprising conjectures after adding auxiliary constructions *of their own initiative*.[8] One example of this kind comes from the Year 2 study where the following problem was presented:

> Divide the sides of a square *ABCD* into three equal parts and form quadrilateral *EGIK* as shown below (see Figure 11).
>
> Compare *ABCD* and *EGIK* with respect to areas, perimeters and lengths of the sides. State your conjectures.

While the problem has an explicit goal, the teacher in this classroom also wanted students to explore their diagrams more broadly and make as many interesting conjectures as they could. One student thought that adding diagonals to the original square might lead to interesting conjectures. After adding the diagonals, she conjectured that they split the sides of the inside quadrilateral into two pieces whose lengths are in a 1 : 2 ratio. She continued



One example

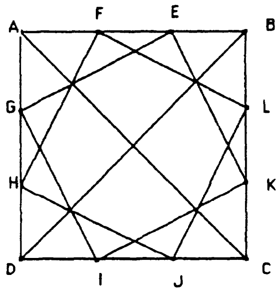


Fig. 11. With an added square and two diagonals.

to add to the construction as shown in Figure 11, and noted that the diagonals pass through the points where the two inside quadrilaterals meet.

Diagrams were not finished products for this student. They could be added to and "continued" by making auxiliary lines. The use of auxiliary lines to form conjectures also transferred to paper and pencil tasks. For example, during Year 5, in a Supposer class of 20 students, for one problem testing conjecturing ability, there were eight different ways of adding auxiliary lines to the given diagram, as shown in Figure 12. Some of the added lines were almost trivial (Fig. 12 top right), while others led

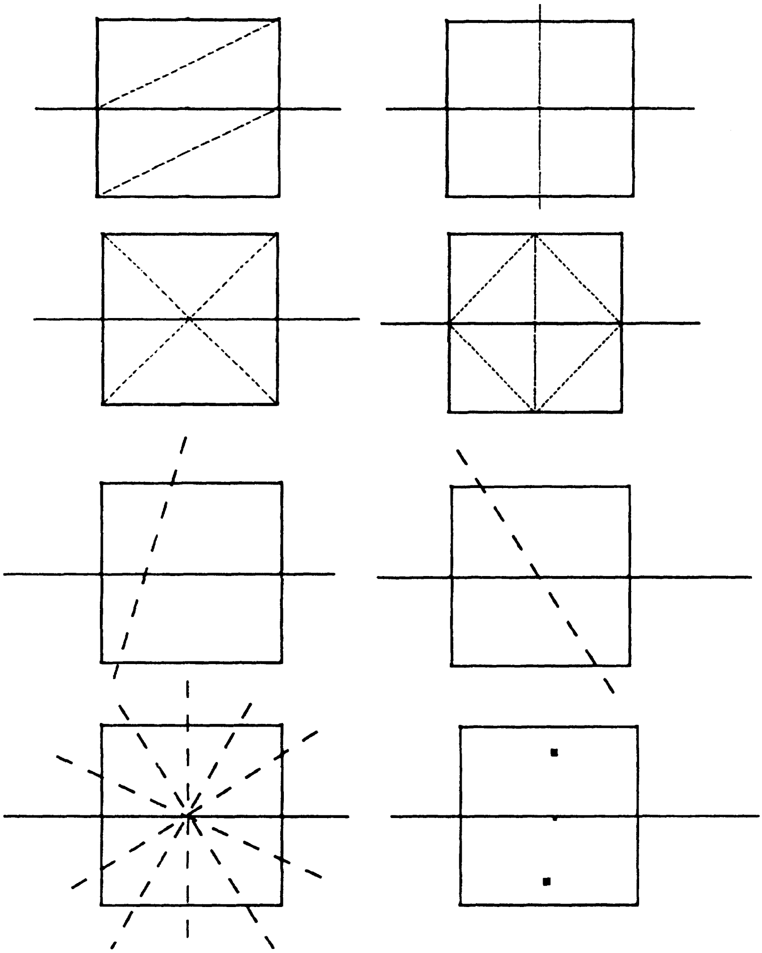


Fig. 12. Eight different auxiliary lines.

to sophisticated ideas like: "There are infinite number of lines going through the center of a square, and as they come closer to the one parallel line their length decreases until it reaches the length of the side of the square" (Fig. 12 bottom left).

SUMMARY AND CONCLUSIONS

Other classes don't go into as much detail as we do... We really go in-depth. Other people, they just know what they (medians) are and what they do. I think we are more familiar with them. (Taken from a year-end interview with a Supposer student reported in Yerushalmy *et al.*, 1987, p. 47.)

I believe that these students did get more out of their Geometry class than they would have done in a traditional class... (Taken from a year-end evaluation by a Supposer teacher reported in Yerushalmy *et al.*, 1987, p. 40.)

When evaluating the effect of an educational innovation, it is important to use appropriate standards, to test for the goals that the innovation has set out to accomplish. One important goal of the Supposer intervention is to help students use diagrams effectively. By providing research evidence, this paper assessed student learning outcomes and evaluated this one aspect of students' geometrical abilities.

The results of this assessment indicate that students using the Supposer understood diagrams and their limitations better than students from the traditional classrooms portrayed in the research literature. They overcame the three diagram-related learning obstacles identified at the beginning of the paper that traditionally beset students, namely, the particularity of diagrams, the tendency to equate standard diagrams with the classes to which they belong, and the inability to view a diagram in different ways. The general approach of the Supposer students in this study to diagrams was flexible, not rigid. Specifically, they were willing and able to focus their attention on different parts of diagrams, look for non-standard diagrams, and add auxiliary lines to diagrams in order to derive conjectures. They were also adept at creating and examining sequences of related diagrams. Their use of these abilities led to many interesting, novel conjectures. Finally, the skills in working with diagrams were also helpful to these students when proving their conjectures. They were able to abstract the variant and invariant aspects of the set of objects under consideration from examining series of diagrams and to use auxiliary lines and their ability to focus on parts of diagrams to derive proofs.

The material presented in this paper also demonstrates that the Supposer students in these studies had acquired important and powerful problem solving strategies for analyzing problems, conjecturing, and proving. One

of the difficulties in analyzing a problem is when the givens and "legal" operations are not specified completely, for example when a diagram which is meant to be a model is provided. Students in this study were adept at analyzing such diagrams and imagining other members of the set which helped them understand the problems' goals. They were not trapped by the particularity of the given diagram. Another skill related to diagrams that enhanced students' ability to analyze problems was the ability to draw a geometrical construction stage by stage observing and analyzing the additional properties of each new element.

The experience reported in this paper and the research literature on visualization make it clear that when creating software tools involving visual representations and intended to aid student inquiry, it is extremely important to understand the obstacles that students experience when working with these representations and design the software to help them overcome these obstacles. The experiences described in this paper also suggest that when such software is designed, students can overcome obstacles to using visual representations and make good use of these representations in their learning and conjecture-making. This conclusion encourages the creation of, and experimentation with, software tools which include a visual representation to aid student conjecturing in other fields of mathematics.

NOTES

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[1] A further indication of the perceived connection between problems of this sort and intellectual maturity is their central role in Feurstein's Instrumental Enrichment program. Examples of this sort of exercise for his students are "Organization of Dots" and "Analytic Perception" (Feurstein *et al.*, 1980). For similar reasons, problems of this sort frequently appear in books like Jim Fixx's *Games for the Superintelligent*.

[2] See Greeno (1983) for an attempt to teach a part/whole reorganization scheme to help students solve one of the problems mentioned by Wertheimer.

[3] The Supposer was written by Judah Schwartz, Michal Yerushalmy, and Education Development Center and is published by Sunburst Communications, Inc. The program allows the user to start with a primitive shape, make a construction, make measurements of the resulting diagram, and repeat the construction on other initial shapes. A more complete description will appear further on in the paper.

[4] It is interesting to note that it is very rare to see a geometry textbook with two diagrams to illustrate a single problem. Indeed, in most texts the situation is reversed, a single diagram is used for two problems.

[5] Other reasons for preferring the language of labels to pointing are related to the goal of helping students conjecture which is not the focus of this paper.

[6] We chose to take all of the remaining examples under this heading from the work of one class to indicate that these phenomena occurred frequently.

[7] See Anderson (1980) for the distinction between taking apart and peeling apart decomposition approaches.
[8] It is important to emphasize once again that the SUPPOSER is only a tool (which in this case makes it easy to draw auxiliary lines). Students' motivation to produce conjectures and to add auxiliary lines in service of this goal is a function of the student and the environment (teacher, classmates etc). In a class which is a community of learners who appreciate good new ideas, students will have reason to use auxiliary lines to make conjectures.

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